Roy’s Identity

The Marshallian demand function **x**(**p**, w) implies Roy’s Identity:

$$x_i(\boldsymbol{p}, w) = -\frac{\frac{\partial V(\boldsymbol{p}, w)}{\partial p_i}}{\frac{\partial V(\boldsymbol{p}, w)}{\partial w}} \ for\ i = 1\ to\ n.$$

Roy’s Identity provides a means of obtaining a demand function from an indirect utility function. Notice that we have the demand function on the left of the equality and we differentiate the indirect utility on the right side with respect to each of its arguments.

Verification of Roy’s Identity: Details

We will start our verification of this equality by working with the numerator on its right side. First recall that changes in utility will be a function of the changes in the consumption of each commodity x_j. Invoking the Chain Rule to find the derivative of indirect utility with respect to each price, we have:

$$\frac{\partial V(\boldsymbol{p}, w)}{\partial p_i} = \frac{\partial U(\boldsymbol{x} * (\boldsymbol{p}, w))}{\partial p_i} = \sum_{j=1}^{n} \frac{\partial U(\boldsymbol{x} *)}{\partial x_j} \frac{\partial x_j}{\partial p_i}$$

Recall from our constrained maximization of U(**p**, w) that:

$$\frac{\partial \mathrm{L}}{\partial \mathrm{x_i}} = \frac{\partial \mathrm{U}(\mathbf{x} *)}{\partial \mathrm{x_i}} - \lambda \mathrm{p_i} = 0 \text{ for i} = 1 \text{ to n}$$

which means that:

$$\frac{\partial \mathrm{U}(\mathbf{x} *)}{\partial \mathrm{x_i}} = \lambda \mathrm{p_i} = 0 \text{ for i} = 1 \text{ to n}$$

and

$$\sum_{j=1}^{n} \frac{\partial U(\boldsymbol{x} *)}{\partial x_j} = \lambda \sum_{j=1}^{n} \mathrm{p_j}$$

Substituting in above, we have:

$$\frac{\partial V(\boldsymbol{p}, w)}{\partial p_i} = \lambda \sum_{j=1}^{n} p_j \frac{\partial x_j}{\partial p_i}$$

This provides with a restatement of the numerator of the right hand side of Roy’s Identity. Next, consider the right side denominator of Roy’s Identity. The Envelope Theorem provides:

$$\frac{\partial V(\boldsymbol{p},w)}{\partial w} = \frac{\partial L(\boldsymbol{x}*,\lambda*)}{\partial w} = \lambda$$

Substituting these into first the denominator and then the numerator of our statement of Roy's Identity leaves:

$$x_i(\boldsymbol{p},w) = -\frac{\lambda \sum_{j=1}^{n} p_j \frac{\partial x_j}{\partial p_i}}{\lambda} = -\sum_{j=1}^{n} p_j \frac{\partial x_j}{\partial p_i} = x_i(\boldsymbol{p},w)\; for\; i = 1\; to\; n.$$

x(**p**, w) defines the consumer's demand function, as we see next from Walra's Law.

Recall that Walra's Law states that total consumption expenditures must equal total wealth:

$$\sum_{j=1}^{n} p_j\, x_j(\boldsymbol{p},w) \equiv w$$

If we differentiate both sides of this with respect to the price p_i of commodity i, we obtain:

$$\frac{\partial \sum_{j=1}^{n} p_j\, x_j(\boldsymbol{p},w)}{\partial p_i} = x_i(\boldsymbol{p},w) + \sum_{j=1}^{n} p_j \frac{\partial x_j}{\partial p_i} = 0 = \frac{\partial w}{\partial p_i}$$

or:

$$x_i(\boldsymbol{p},w) = -\sum_{j=1}^{n} p_j \frac{\partial x_j}{\partial p_i}$$

This is our demand function.

Roy's Identity, enables us to derive demand functions from the indirect utility functions. In many cases this will be easier than directly estimating demand functions **x**(**p**, w). Estimating Roy's Identity requires estimation of a single equation while estimation of **x**(**p**, w) might require an estimate of each value for p and w the solution to a set of n+1 first-order equations. This will be applied in our derivation of the Slutsky Equation later.